

FAITHFULNESS OF BI-FREE PRODUCT STATES

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ABSTRACT. Given a non-trivial family of pairs of faces of unital C^* -algebras where each pair has a faithful state, it is proved that if the bi-free product state is faithful on the reduced bi-free product of this family of pairs of faces then each pair of faces arises as a minimal tensor product. A partial converse is also obtained.

1. INTRODUCTION

The reduced free product was given independently by Avitzour [1] and Voiculescu [7] and it has been foundational in the development of free probability. Dykema proved in [2] that the free product state on the reduced free product of unital C^* -algebras with faithful states is faithful. In consequence of this, if $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ is a free family of unital C^* -algebras in the non-commutative C^* -probability space (\mathcal{A}, φ) and if φ is faithful on $C^*(\{\mathcal{A}_i\}_{i \in \mathcal{I}})$ then

$$C^*(\{\mathcal{A}_i\}_{i \in \mathcal{I}}) \simeq *_{i \in \mathcal{I}}(\mathcal{A}_i, \varphi|_{\mathcal{A}_i}),$$

the reduced free product of the \mathcal{A}_i 's with respect to the given states. This can be deduced from a paper of Dykema and Rørdam, namely [3, Lemma 1.3].

The present paper is the result of the author's attempt to prove the same result in the new context of bi-free probability introduced by Voiculescu [8]. To this end, suppose $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ is a non-trivial family of pairs of faces in the non-commutative C^* -probability space (\mathcal{A}, φ) . If $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$ is faithful on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$, for all $i \in \mathcal{I}$, then it will be proven that if the bi-free product state $**_{i \in \mathcal{I}} \varphi_i$ is faithful on the reduced bi-free product $**_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ then $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}, i \in \mathcal{I}$. A converse is shown with the added assumption that each φ_i is a product state. Moreover, in this case there is a commensurate result to that which follows from Dykema and Rørdam, mentioned above.

It should be mentioned that the failure in general of the faithfulness of the bi-free product state has been pointed out in [4] and this failure has been the cause of the introduction of weaker versions of faithfulness in the bi-free context [4, 5].

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2. BI-FREE INDEPENDENCE AND THE REDUCED BI-FREE PRODUCT

We will first take some time to recall the definition of bi-free independence from [8] and then define the reduced bi-free product of C^* -algebras and the bi-free product state.

Fix a non-commutative C^* -probability space (\mathcal{A}, φ) , that is a unital C^* -algebra and a state. Given a set \mathcal{I} , suppose that for each $i \in \mathcal{I}$ there is a pair of unital C^* -subalgebras $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ of \mathcal{A} , a “left” algebra and a “right” algebra. We call the set $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ a family of pairs of faces in \mathcal{A} . Such a family will be called *non-trivial* if $|\mathcal{I}| \geq 2$ and $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \neq \mathbb{C}$ for all $i \in \mathcal{I}$, there are no trivial pairs of faces.

Let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS construction for $(C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$ where $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$. Voiculescu [8] (and even way back in [7]) observed that there are two natural representations of $B(\mathcal{H}_i)$ on $B(\mathcal{H})$, namely the left representation $\lambda_i : B(\mathcal{H}_i) \rightarrow B(\mathcal{H})$ and the right representation $\rho_i : B(\mathcal{H}_i) \rightarrow B(\mathcal{H})$, where

$$(\mathcal{H}, \xi) = *_{i \in \mathcal{I}} (\mathcal{H}_i, \xi_i)$$

is the Hilbert space free product. Recall that, if $\mathring{\mathcal{H}}_j = \mathcal{H}_j \ominus \mathbb{C}\xi_j$, then

$$\mathcal{H} := \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ i_1, \dots, i_n \in \mathcal{I} \\ i_1 \neq \dots \neq i_n}} \mathring{\mathcal{H}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{H}}_{i_n}$$

To define these representations we need to first build some Hilbert spaces and some unitaries. To this end, define

$$\begin{aligned} \mathcal{H}(l, i) &:= \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ i_1, \dots, i_n \in \mathcal{I} \\ i \neq i_1 \neq \dots \neq i_n}} \mathring{\mathcal{H}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{H}}_{i_n} \quad \text{and} \\ \mathcal{H}(r, i) &:= \mathbb{C}\xi \oplus \bigoplus_{\substack{n \in \mathbb{N} \\ i_1, \dots, i_n \in \mathcal{I} \\ i_1 \neq \dots \neq i_n \neq i}} \mathring{\mathcal{H}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{H}}_{i_n}. \end{aligned}$$

Then there are unitaries $V_i : \mathcal{H}_i \otimes \mathcal{H}(l, i) \rightarrow \mathcal{H}$ and $W_i : \mathcal{H}(r, i) \otimes \mathcal{H}_i \rightarrow \mathcal{H}$ given by concatenation (with appropriate handling of ξ_i and ξ). Now define the left and right representations,

$$\lambda_i(T) = V_i(T \otimes I_{\mathcal{H}(l, i)})V_i^* \quad \text{and} \quad \rho_i(T) = W_i(I_{\mathcal{H}(r, i)} \otimes T)W_i^*.$$

With all of this groundwork established we can finally define bi-free independence. Note that $*$ below refers to the full (or universal) free product of C^* -algebras.

Definition 2.1 (Voiculescu [8]). The family of pairs of faces $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ is said to be *bi-freely independent* with respect to φ if the following diagram commutes

$$\begin{array}{ccccc} \check{*}_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)}) & \xrightarrow{\iota} & C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}) & \xrightarrow{\varphi} & \mathbb{C} \\ \downarrow \check{*}_{i \in \mathcal{I}}(\pi_i \check{*} \pi_i) & & & & \parallel \\ \check{*}_{i \in \mathcal{I}}(B(\mathcal{H}_i) \check{*} B(\mathcal{H}_i)) & \xrightarrow{\check{*}_{i \in \mathcal{I}}(\lambda_i \check{*} \rho_i)} & B(\mathcal{H}) & \xrightarrow{\langle \cdot, \xi \rangle} & \mathbb{C} \end{array}$$

where ι is the unique $*$ -homomorphism extending the identity on each $\mathcal{A}_\chi^{(i)}$, for all $\chi \in \{l, r\}$ and $i \in \mathcal{I}$.

From this we can now define the main objects of this paper.

Definition 2.2. Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ be a family of pairs of faces in the non-commutative C^* -probability space (\mathcal{A}, φ) . As before, denote φ_i to be the restriction of φ to $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ and let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS construction of $(C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$.

The *reduced bi-free product* of $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ with respect to the states φ_i is

$$(**_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), **_{i \in \mathcal{I}}\varphi_i) = **_{i \in \mathcal{I}}((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$$

which is made up of the unital C^* -subalgebra of $B(\mathcal{H})$, called the *reduced bi-free product of C^* -algebras*,

$$**_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) := C^*((\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i \in \mathcal{I}}) \subset B(\mathcal{H})$$

and the *bi-free product state*

$$**_{i \in \mathcal{I}}\varphi_i(\cdot) := \langle \cdot, \xi \rangle.$$

It is an immediate fact that the family of pairs of faces $(\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i \in \mathcal{I}}$ is bi-freely independent with respect to the bi-free product state.

It should be noted that we are working within the framework of the original non-commutative C^* -probability space (\mathcal{A}, φ) . This means that the reduced bi-free product is taking into account the behaviour of φ not just on the left and right faces but on the C^* -algebra they generate, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$. Since bi-free independence is a statement about the behaviour in the original C^* -probability space this definition makes sense.

That being said, one can start with faces in different C^* -probability spaces and simply create a new C^* -probability space by taking the full free product of the C^* -algebras and their associated states and then proceed with the above reduced bi-free product construction.

3. FAITHFULNESS OF BI-FREE PRODUCT STATES

We first establish what happens when the bi-free product state is faithful.

Theorem 3.1. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ be a non-trivial family of pairs of faces in the non-commutative C^* -probability space (\mathcal{A}, φ) such that $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$ is faithful on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ for each $i \in \mathcal{I}$. If $**_{i \in \mathcal{I}} \varphi_i$ is faithful on the reduced bi-free product $**_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ then*

$$C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}.$$

Proof. As usual we will be using the notation from Section 2. First we will establish that $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute in \mathcal{A} , then we will show that they induce a C^* -norm on the algebraic tensor product $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ and finally that this is in fact the minimal tensor norm.

To simplify the notation a little bit, because the φ_i are assumed to be faithful, consider $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ as already a subalgebra of $B(\mathcal{H}_i)$ and so $\varphi_i(\cdot) = \langle \cdot \xi_i, \xi_i \rangle$. That is, we are suppressing the π_i notation from the GNS construction.

Suppose $a_\chi \in \mathcal{A}_\chi^{(i)}$ such that $\varphi_i(a_\chi) = 0$, $\chi \in \{l, r\}$ and $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$. Such a b exists by the non-triviality of the family of pairs of faces. Since $b \neq 0$ and $\langle b\xi, \xi \rangle = 0$ there exists an $h \in \mathring{\mathcal{H}}_j$ such that $\langle bh, \xi_j \rangle \neq 0$. Otherwise, $\langle b^*\xi_j, h \rangle = 0$ for all $h \in \mathring{\mathcal{H}}_j \oplus \mathbb{C}\xi_j$, that is $b^*\xi_j = 0$ which by faithfulness, since ξ_j is separating, implies that $b^* = 0$ and so $b = 0$.

Now, [8, Section 1.5] establishes that $[\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)})](\mathcal{H} \ominus \mathcal{H}_i) = 0$ which gives that

$$(\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b))\xi = 0$$

since $b\xi \in \mathring{\mathcal{H}}_j \subset \mathcal{H}$. The faithfulness of $**_{i \in \mathcal{I}} \varphi_i$ implies that ξ is a separating vector for the reduced bi-free product and thus

$$\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b) = 0$$

which gives that

$$\begin{aligned} 0 &= P_{\mathcal{H}_i}(\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b))h \\ &= (\lambda_i(a_l)\rho_i(a_r) - \rho_i(a_r)\lambda_i(a_l))\langle bh, \xi_j \rangle \xi \\ &= \langle bh, \xi_j \rangle (a_l a_r - a_r a_l) \xi_i. \end{aligned}$$

Since ξ_i is separating for $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ this implies that a_l and a_r commute. Thus, $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute in \mathcal{A} for every $i \in \mathcal{I}$.

Claim: $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ is faithfully contained in $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$.

Since $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute we simply need to show that the $*$ -homomorphism

$$\sum_{k=1}^m a_{k,l} \odot a_{k,r} \mapsto \sum_{k=1}^m a_{k,l} a_{k,r}$$

is injective. To this end, consider $h \in \mathcal{H}_j$, $\|h\| = 1$ where $j \neq i$ and the isometric map

$$V_h : \mathcal{H}_i \otimes \mathcal{H}_i \rightarrow \mathcal{H}_i \otimes h \otimes \mathcal{H}_i$$

defined by $V_h(h_l \otimes h_r) = h_l \otimes h \otimes h_r$ for $h_l, h_r \in \mathcal{H}_i$. Note that in \mathcal{H} we really have that

$$\mathcal{H}_i \otimes h \otimes \mathcal{H}_i = \mathbb{C}h \oplus (\overset{\circ}{\mathcal{H}}_i \otimes h) \oplus (h \otimes \overset{\circ}{\mathcal{H}}_i) \oplus (\overset{\circ}{\mathcal{H}}_i \otimes h \otimes \overset{\circ}{\mathcal{H}}_i)$$

but hopefully the reader will pardon the simplified notation.

Now $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ is a reducing subspace of $C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$ since for all $a \in \mathcal{A}_l^{(i)}$, $b \in \mathcal{A}_r^{(i)}$ and $\eta_1, \eta_2 \in \mathcal{H}_i$ we have that

$$\begin{aligned} V_h^* \lambda_i(a) \rho_i(b) V_h(\eta_1 \otimes \eta_2) &= V_h^* \lambda_i(a) \rho_i(b)(\eta_1 \otimes h \otimes \eta_2) \\ &= a \eta_1 \otimes b \eta_2. \end{aligned}$$

Thus, compressing to $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ gives

$$V_h^* C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)})) V_h = \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}.$$

So, if $\sum_{k=1}^m a_{k,l} \odot a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ then $\sum_{k=1}^m a_{k,l} \otimes a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}$ which implies that

$$\begin{aligned} 0 &\neq \sum_{k=1}^m a_{k,l} \otimes a_{k,r} (\xi_i \otimes \xi_i) \\ &= V_h^* \sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) V_h(\xi_i \otimes \xi_i) \\ &= \sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) h \end{aligned}$$

since the state $\langle \xi_i \otimes \xi_i, \xi_i \otimes \xi_i \rangle$ is faithful on the min tensor product. But then $\sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \neq 0 \in C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$ which by the faithfulness of $**_{i \in \mathcal{I}} \varphi_i$ gives that $\sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \xi \neq 0$. Finally,

$$\begin{aligned} 0 &\neq \left\langle \sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \xi, \sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \xi \right\rangle \\ &= \left\langle \left(\sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \right)^* \left(\sum_{k=1}^m \lambda_i(a_{k,l}) \rho_i(a_{k,r}) \right) \xi, \xi \right\rangle \\ &= \varphi_i \left(\left(\sum_{k=1}^m a_{k,l} a_{k,r} \right)^* \left(\sum_{k=1}^m a_{k,l} a_{k,r} \right) \right) \end{aligned}$$

which gives by the faithfulness of φ_i that $\sum_{k=1}^m a_{k,l} a_{k,r} \neq 0$. Therefore, the claim is verified.

Now, this implies that $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_\alpha \mathcal{A}_r^{(i)}$ where $\|\cdot\|_\alpha$ is a C^* -norm on $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$. So by Takesaki's Theorem [6] we have that there exists a surjective $*$ -homomorphism

$$q : C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \rightarrow \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}.$$

To finish the proof all we need to do is show that q is injective.

To this end, let $a \in C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $q(a) = 0$. Again as in the first part of this proof, find $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$ and $h \in \mathcal{H}_j$ such that $\langle bh, \xi_j \rangle \neq 0$. Additionally, assume that $\|b\xi_j\| = 1$.

In the second part of this proof we saw that compressing to $\mathcal{H}_i \otimes b\xi_j \otimes \mathcal{H}_i$ is tantamount to this quotient homomorphism q . Namely, suppose $\iota_i : \mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)} \rightarrow C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ is the unique $*$ -homomorphism extending the identity in each component. There then exists $\tilde{a} \in \mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)}$ such that $\iota_i(\tilde{a}) = a$. An important fact to record is that $\lambda_i * \rho_i(\cdot)|_{\mathcal{H}_i} = \iota_i(\cdot)$. Thus,

$$V_{b\xi_j}^* \lambda_i * \rho_i(\tilde{a}) V_{b\xi_j} = q(a) = 0,$$

which implies, by the fact that $V_{b\xi_j}(\mathcal{H}_i \otimes \mathcal{H}_i)$ is reducing for $\lambda_i * \rho_i(\mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)})$, that

$$\begin{aligned} 0 &= \lambda_i * \rho_i(\tilde{a}) V_{b\xi_j}(\xi_i \otimes \xi_i) \\ &= \lambda_i * \rho_i(\tilde{a})(b\xi_j) \\ &= \lambda_i * \rho_i(\tilde{a}) \lambda_j * \rho_j(b) \xi. \end{aligned}$$

By the faithfulness of the bi-free product state $\lambda_i * \rho_i(\tilde{a}) \lambda_j * \rho_j(b) = 0$ and so

$$\begin{aligned} 0 &= P_{\mathcal{H}_i} \lambda_i * \rho_i(\tilde{a}) \lambda_j * \rho_j(b) h \\ &= \lambda_i * \rho_i(\tilde{a}) \langle bh, \xi_j \rangle \xi \\ &\quad \langle bh, \xi_j \rangle \iota_i(\tilde{a}) \xi \\ &\quad \langle bh, \xi_j \rangle a \xi_i. \end{aligned}$$

Hence, by the faithfulness of φ_i we have that $a = 0$. Therefore, for all $i \in \mathcal{I}$, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}$. \square

We turn now to a partial converse of the previous theorem.

Theorem 3.2. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ be a family of pairs of faces in the non-commutative C^* -probability space (\mathcal{A}, φ) . If $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}$ and $\varphi_i = \varphi_i|_{\mathcal{A}_l^{(i)}} \otimes \varphi_i|_{\mathcal{A}_r^{(i)}}$ is a faithful product state on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$, for all $i \in \mathcal{I}$, then $**_{i \in \mathcal{I}} \varphi_i$ is faithful on the reduced bi-free product and*

$$**_{i \in \mathcal{I}} (\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}} \simeq *_{i \in \mathcal{I}} (\mathcal{A}_l^{(i)}, \varphi) \otimes_{\min} *_{i \in \mathcal{I}} (\mathcal{A}_r^{(i)}, \varphi).$$

Proof. As before, we will be using the notation of Section 2.

For each $i \in \mathcal{I}$, since $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}$ and φ_i is a product state we can a priori choose $\mathcal{H}_i = \mathcal{H}_{i,l} \otimes \mathcal{H}_{i,r}$, unit vectors $\xi_{i,l} \in \mathcal{H}_{i,l}, \xi_{i,r} \in \mathcal{H}_{i,r}$ such that $\xi_i = \xi_{i,l} \otimes \xi_{i,r}$ and $*$ -homomorphisms $\pi_{i,\chi} : \mathcal{A}_\chi^{(i)} \rightarrow B(\mathcal{H}_{i,\chi})$ such that $\pi_i = \pi_{i,l} \otimes \pi_{i,r}$. This will give for $a_\chi \in \mathcal{A}_\chi^{(i)}, \chi \in \{l, r\}$, that

$$\begin{aligned} \varphi_i(a_l a_r) &= \langle \pi_i(a_l a_r) \xi_i, \xi_i \rangle \\ &= \langle \pi_{i,l}(a_l) \xi_{i,l}, \xi_{i,l} \rangle \langle \pi_{i,r}(a_r) \xi_{i,r}, \xi_{i,r} \rangle. \end{aligned}$$

Define for $\chi \in \{l, r\}$ the Hilbert space free products

$$(\mathcal{H}_\chi, \xi_\chi) = *_{i \in \mathcal{I}} (\mathcal{H}_{i,\chi}, \xi_{i,\chi}).$$

Since there are multiple Hilbert space free products we will use subscripts to denote the different left and right representations. Specifically, $\lambda_{\mathcal{H}_i}, \lambda_{\mathcal{H}_{i,l}}, \rho_{\mathcal{H}_i}$ and $\rho_{\mathcal{H}_{i,r}}$.

Dykema's original result [2] proves that $\langle \cdot, \xi_\chi \rangle$ is faithful on $*_{i \in \mathcal{I}} (\mathcal{A}_\chi^{(i)}, \varphi)$ for $\chi \in \{l, r\}$ and it is a folklore result that the minimal tensor product of faithful states is faithful. Thus, $\langle \cdot, \xi_l \otimes \xi_r, \xi_l \otimes \xi_r \rangle$ is faithful on $*_{i \in \mathcal{I}} (\mathcal{A}_l^{(i)}, \varphi) \otimes_{\min} *_{i \in \mathcal{I}} (\mathcal{A}_r^{(i)}, \varphi)$.

Fix $k \geq 1$ and $j_1, \dots, j_k \in \mathcal{I}$ such that $j_i \neq j_{i+1}, 1 \leq i \leq k-1$. Now fix a unit vector

$$\begin{aligned} h &= (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \dots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}) \\ &\in (\xi_{j_1,l} \otimes \mathring{\mathcal{H}}_{j_1,r}) \otimes \mathring{\mathcal{H}}_{j_2} \otimes \dots \otimes \mathring{\mathcal{H}}_{j_{k-1}} \otimes (\mathring{\mathcal{H}}_{j_k,l} \otimes \xi_{j_k,r}). \end{aligned}$$

Call the collection of such h , as k and the indices vary, $\mathcal{S} \subset \mathcal{H}$.

We need to make a small aside to study this subset \mathcal{S} . Suppose $m \geq 1, s_1, \dots, s_m \in \mathcal{I}$ such that $s_t \neq s_{t+1}$ for $1 \leq t \leq m-1$, and $\eta_{t,l} \in \mathcal{H}_{s_t,l}, \eta_{t,r} \in \mathcal{H}_{s_t,r}$ but $\eta_{t,l} \otimes \eta_{t,r} \in \mathring{\mathcal{H}}_{s_t}$ for $1 \leq t \leq m$. This last condition implies that $\eta_{t,l}$ and $\eta_{t,r}$ cannot both be $\xi_{t,\chi}$ for $\chi \in \{l, r\}$. It can be established that there exists $h \in \mathcal{S}$ such that

$$\eta := (\eta_{1,l} \otimes \eta_{1,r}) \otimes \dots \otimes (\eta_{m,l} \otimes \eta_{m,r}) \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r.$$

In particular, let $0 \leq v \leq m$ be the biggest number such that $\eta_{t,r} \in \mathbb{C}\xi_{s_t,r}, 1 \leq t \leq v$ and let $1 \leq w \leq m+1$ be the smallest number such that $\eta_{t,l} \in \mathbb{C}\xi_{s_t,l}, w \leq t \leq m$. By the condition on the non-triviality of the $\eta_{t,\chi}$ we have that $v < w$. If $v = m$ then $w = m+1$ and $\eta \in \mathcal{H}_l$, and if $w = 1$ then $v = 0$ and $\eta \in \mathcal{H}_r$. Otherwise, when $0 \leq v \leq m-1$ and $2 \leq w \leq m$, define

$$\begin{aligned} \eta_l &= (\eta_{1,l} \otimes \|\eta_{1,r}\| \xi_{s_1,r}) \otimes \dots \otimes (\eta_{v,l} \otimes \|\eta_{v,r}\| \xi_{s_v,r}) \otimes (\eta_{v+1,l} \otimes \xi_{s_{v+1},r}), \\ \eta_S &= (\xi_{v+1,l} \otimes \eta_{v+1,r}) \otimes \dots \otimes (\eta_{w-1,l} \otimes \xi_{s_{w-1},r}), \\ \eta_r &= (\xi_{w-1,l} \otimes \eta_{w-1,r}) \otimes (\|\eta_{w,l}\| \xi_{s_w,l} \otimes \eta_{w,r}) \otimes \dots \otimes (\|\eta_{m,l}\| \xi_{s_m,l} \otimes \eta_{m,r}) \end{aligned}$$

with $\eta_S = \xi$ if $v+1 = w$. Hence, by the usual slight abuse of the tensor notation, $\eta = \eta_l \otimes \eta_S \otimes \eta_r \in \mathcal{H}_l \otimes \eta_S \otimes \mathcal{H}_r$ with $\eta_S \in \mathcal{S}$. Therefore,

$$\overline{\text{span}}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}$$

since η was an arbitrary centered elementary tensor.

For any $h \in \mathcal{S}$ there is then a natural isometric map $S_h : \mathcal{H}_l \otimes \mathcal{H}_r \rightarrow \mathcal{H}$ given by the concatenation $\mathcal{H}_l \otimes \mathcal{H}_r \mapsto \mathcal{H}_l \otimes h \otimes \mathcal{H}_r$ with the appropriate simplification of tensors when needed. In particular,

$$\begin{aligned} \xi_l \otimes \xi_r &\mapsto h, \\ \xi_l \otimes (\mathring{\mathcal{H}}_{i_1,r} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,r}) &\mapsto \\ \begin{cases} h \otimes (\xi_{i_1,l} \otimes \mathring{\mathcal{H}}_{i_1,r}) \otimes \cdots \otimes (\xi_{i_m,l} \otimes \mathring{\mathcal{H}}_{i_m,r}), & i_1 \neq j_k \\ (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \mathring{\mathcal{H}}_{i_1,r}) \otimes \\ \cdots \otimes (\xi_{i_m,l} \otimes \mathring{\mathcal{H}}_{i_m,r}), & i_1 = j_k \end{cases} \\ (\mathring{\mathcal{H}}_{i_1,l} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,l}) \otimes \xi_r &\mapsto \\ \begin{cases} (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h, & i_m \neq j_1 \\ (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \\ \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}), & i_m = j_1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} (\mathring{\mathcal{H}}_{i_1,l} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,l}) \otimes (\mathring{\mathcal{H}}_{t_1,r} \otimes \cdots \otimes \mathring{\mathcal{H}}_{t_s,r}) &\mapsto \\ (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h \otimes (\xi_{t_1,l} \otimes \mathring{\mathcal{H}}_{t_1,r}) \otimes \cdots \otimes (\xi_{t_s,l} \otimes \mathring{\mathcal{H}}_{t_s,r}) \end{aligned}$$

if $i_m \neq j_1$ and $j_k \neq t_1$ with similar statements as the cases above when $i_m = j_1$ or $j_k = t_1$ or both happen. Perhaps the most natural case of S_h is when $h = \xi$. It certainly minimizes, but doesn't remove, the need for all of the cases above.

A careful examination of the S_h isometric map implies that for $a \in \mathcal{A}_l^{(i_1)}$, $b \in \mathcal{A}_r^{(i_2)}$ and $\eta_\chi \in \mathcal{H}_\chi$ for $\chi \in \{l, r\}$ we have that, by abuse of the tensor notation,

$$\begin{aligned} \lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))S_h(\eta_l \otimes \eta_r) &= \lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))(\eta_l \otimes h \otimes \eta_r) \\ &= \lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes h \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r \\ &= S_h(\lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r) \end{aligned}$$

Hence, $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$ is a reducing subspace of the reduced bi-free product. Moreover,

$$S_h^* \lambda_{\mathcal{H}_i} \circ \pi_i(\cdot) S_h = (\lambda_{\mathcal{H}_{i,l}} \circ \pi_{i,l}(\cdot)) \otimes I_{\mathcal{H}_r} \quad \text{on } \mathcal{A}_l^{(i)}$$

and

$$S_h^* \rho_{\mathcal{H}_i} \circ \pi_i(\cdot) S_h = I_{\mathcal{H}_l} \otimes (\rho_{\mathcal{H}_{i,r}} \circ \pi_{i,r}(\cdot)) \quad \text{on } \mathcal{A}_r^{(i)}$$

Therefore, for any $h \in \mathcal{S}$,

$$S_h^* (**_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})) S_h = **_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{\min} **_{i \in \mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi)$$

and furthermore, by the identities involving S_h, λ and ρ , $S_h^* a S_h = S_\xi^*(a) S_\xi$ for all $h \in \mathcal{S}$ and $a \in **_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$.

Finally, we want to show that compression to $S_\xi(\mathcal{H}_l \otimes \mathcal{H}_r)$ is a $*$ -isomorphism. Note that this is the same as compression to $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$ being injective for any $h \in \mathcal{S}$. This gives us a way forward. Suppose that $a \in **_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $**_{i \in \mathcal{I}}\varphi_i(a^*a) = 0$. This implies that

$$\begin{aligned} 0 &= a\xi \\ &= S_\xi^* a S_\xi(\xi_l \otimes \xi_r). \end{aligned}$$

By the faithfulness of $\langle \cdot, \xi_l \otimes \xi_r, \xi_l \otimes \xi_r \rangle$ this gives that $S_\xi^* a S_\xi = 0$ or rather a is 0 on the reducing subspace $S_\xi(\mathcal{H}_l \otimes \mathcal{H}_r)$. But then for all $h \in \mathcal{S}$ we have that

$$S_h^* a S_h = S_\xi^* a S_\xi = 0$$

and a is 0 on the reducing subspace $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$. By what we proved about the set \mathcal{S} , we have that a is 0 on

$$\overline{\text{span}}\{S_h(\mathcal{H}_l \otimes \mathcal{H}_r) : h \in \mathcal{S}\} = \overline{\text{span}}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}.$$

Therefore, $a = 0$ and thus $**_{i \in \mathcal{I}}\varphi_i$ is faithful. \square

There may exist a full converse to Theorem 3.1 but the previous proof highly depends on the state φ_i arising as a tensor product of states. In general, φ_i need not be of this form. We should note here that if $\varphi_i|_{\mathcal{A}_l^{(i)}}$ or $\varphi_i|_{\mathcal{A}_r^{(i)}}$ is a pure state then φ_i will be a tensor product of states.

To end this paper, we summarize with the following corollary.

Corollary 3.3. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ be a non-trivial family of pairs of faces in the non-commutative C^* -probability space (\mathcal{A}, φ) . If φ is faithful on $C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}})$, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{\min} \mathcal{A}_r^{(i)}$, $\varphi_i = \varphi_i|_{\mathcal{A}_l^{(i)}} \otimes \varphi_i|_{\mathcal{A}_r^{(i)}}$ and $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}$ is bi-freely independent with respect to φ , then*

$$\begin{aligned} C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}) &\simeq **_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}} \\ &\simeq *_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{\min} *_{i \in \mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi). \end{aligned}$$

Proof. Recall, that by bi-free independence we know that the following diagram commutes

$$\begin{array}{ccccc} **_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) & \xrightarrow{\iota} & C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i \in \mathcal{I}}) & \xrightarrow{\varphi} & \mathbb{C} \\ \downarrow *_{i \in \mathcal{I}}(\pi_i * \pi_i) & & & & \parallel \\ **_{i \in \mathcal{I}}(B(\mathcal{H}_i), B(\mathcal{H}_i)) & \xrightarrow{*_{i \in \mathcal{I}}(\lambda_i * \rho_i)} & B(\mathcal{H}) & \xrightarrow{\langle \xi, \xi \rangle} & \mathbb{C} \end{array}$$

Because both of the states are faithful on their algebras then for any $a^*a \in **_{i \in \mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$, a^*a is in the kernel of ι if and only if a^*a is in the kernel of $*_{i \in \mathcal{I}}(\lambda_i * \rho_i) \circ *_{i \in \mathcal{I}}(\pi_i * \pi_i)$. Therefore, both quotients are $*$ -isomorphic and Theorem 3.2 gives the final $*$ -isomorphism. \square

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